

Going to use a toy model of species counterpoint for the purposes of explanation:

- Perfect consonances are fifth (4)¹ and octave (7).
- Imperfect consonances are third (2), fourth (3), and sixth (5).
- Perfect consonances must be approached by contrary or oblique motion.
- No restriction on melodic intervals.
- No unisons or crossing of voices.
- Harmonic intervals must be perfect or imperfect.

Here's an elementary example of 3-part species counterpoint:

Here are three melodies that can be transposed so that any two of them can be played with each other:

We can put these together using a modified version of the 3-part notation:

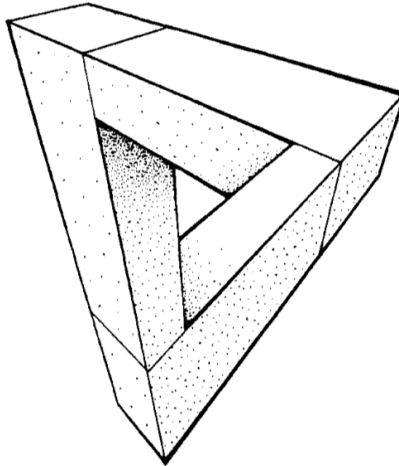
¹I'm numbering intervals counting from 0 (=unison).

But the interval numbers no longer add up vertically! But they do have $bottom + medium - 1 = top$. Call -1 the obstruction. To work out if the counterpoint is correct, you just need to know the intervals and melodic steps.

To write counterpoint of this form by hand, you use the following method:

- Pick a source melody to harmonize.
- Pick an obstruction.
- Write out all the impossible triad shapes for this obstruction².
- Fill in the rest of the counterpoint.

This is an impossible to perform composition in the same way that the Penrose triangle is impossible to realize³:

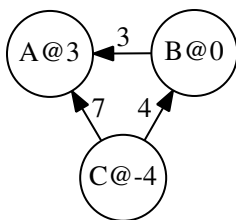


If any two pieces (melodies) are present, the figure(counterpoint) can be realized, but all three parts cannot be combined together at once.

Take the first example given above of a vanilla counterpoint exercise. We can represent the starting pitches using integers - melody A starts on f , which we'll call 3 ($c = 0$, $d = 1$, &c.), so let's notate it as $A@3$. And let's label the edges with intervals:

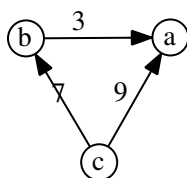
²I made a page here that generates them for you : <https://ded.increpare.com/~locus/comusic/chordgenerate.html>.

³cf Penrose's paper for the geometrical analogue of what is to follow : <http://www.iri.upc.edu/people/ros/StructuralTopology/ST17/st17-05-a2-ocr.pdf>.

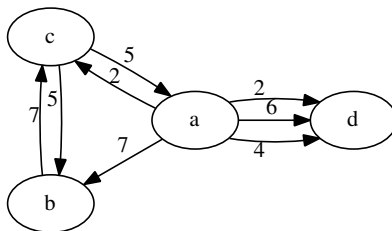


Note that if you travel in a loop and add up the numbers as you go⁴, you get $3 + 4 - 7 = 0$. It adds up to zero.

For the impossible example, we no longer have fixed base positions, so we don't label the vertices, but we can still label the edges, to get this:



Note that the triangle no longer adds to 0, but to 1^5 . Here's a graph of a more complicated system of voice interrelationships:



Looking at this graph, the edges tell you what voices can be harmonized with what other voices, and if you want 3 at once, you're looking for triangles where the edges sum to zero⁶.

Here's the semi-precise, insufficiently-explained analogy with manifolds and cohomological language:

⁴negating them if they're on arrows facing the opposite direction

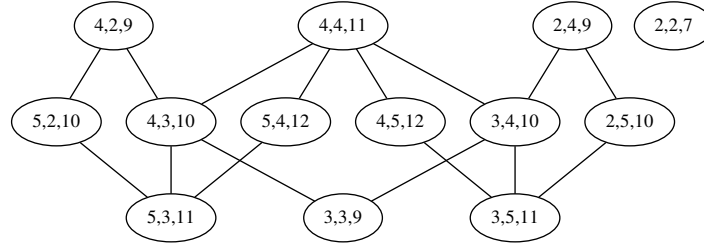
⁵or -1, whatever.

⁶This relates to a the fact that if all circuits sum to zero, you can assign numbers to the vertices such that an edge's value is the difference of its two vertices. This is why the obstruction can be called an obstruction: its presence stops you from realizing a graph with nicely-labeled vertices.

Manifold	Čech Cohomology	Counterpoint System
atlas chart transition area	vertex edges nerve 1-cocycle 1-coboundary 0-cocycle obstruction	set of melodies melody counterpoint pair graph of melodies+intervals interval choices compatible intervals assignment of pitches obstruction

Definition $H(m_1, m_2, i)$ express the proposition “ m_1 harmonizes with m_2 when at interval i .”

For a given model of counterpoint, and a given obstruction, we can construct parallel theories of voice leading. Here’s something for a system with obstruction 3, showing ways of moving from one triad shape another by changing a single note by a single step⁷:



While it’s possible (and entertaining) to just go and write 3-staff counterpoints by hand, but here’s a more algorithmic approach:

- Start with a set M of melodies.
- Generate a graph G with a set of vertices M , with labeled edges:

$$E = \{(m_1 \xrightarrow{i} m_2) | m_1, m_2 \in M, i \in \mathbb{Z}, H(m_1, m_2, i)\}$$

You can view this as a directed graph with a 1-cocycle.

- Complete subgraphs of G in which all triangles sum to 0 are realizable counterpoints.

<https://ded.increpare.com/~locus/comusic/graphgenerate.html> is an implementation of this algorithm. And here’s one that does the same, but searches through canons instead: <https://ded.increpare.com/~locus/comusic/canongenerate.html>.

⁷Here’s a program for calculating them : <https://ded.increpare.com/~locus/comusic/leadingsgenerate.html>.